(0, 1)-MATRICES WITH MINIMAL PERMANENTS

BY

HENRYK MINC†

ABSTRACT

It is shown that the permanent of a totally indecomposable (0,1)-matrix is equal to its largest row sum if and only if all its other row sums are 2.

An n-square $(0,1)$-matrix is said to be *partly decomposable* if it contains an $s \times (n-s)$ zero submatrix; otherwise it is *totally* (or *fully*) *indecomposable.* A totally indecomposable $(0,1)$-matrix is called *nearly decomposable* if the replacement of any of its positive entries by 0 renders it partly decomposable, i.e., if for every positive entry a_{hk} the matrix $A - E_{hk}$ is partly decomposable (E_{hk} denotes the n-square matrix with 1 in the (h,k) position and zeros elsewhere). Let $A(i|j)$ denote the $(n-1)$-square submatrix obtained from A by deleting its ith row and jth column, and let r_i denote the ith row sum of $A = (a_{ij})$,

$$r_i = \sum_{j=1}^{n} a_{ij}, \qquad i = 1, \cdots, n.$$

The main result of this note concerns the case of equality in the following theorem.

THEOREM. *If A is a totally indecomposable $(0,1)$-matrix with row sums $r_1, \cdots, r_n$, then*

$$\text{per}(A) \geqq \max_i r_i. \tag{1}$$

Equality can hold in (1) *if and only if at least $n-1$ of the row sums are* 2.

There is, of course, an exact analogue of the theorem involving column sums instead of row sums.

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Note that the condition for equality does not determine the zero pattern of the matrix (not even modulo permutations of rows and columns) nor, in particular, its column sums.

In order to prove the theorem we require the following three known results.

LEMMA 1 (Minc [3]). *If A is a totally indecomposable* (0, 1)-*matrix then*

$$\operatorname{per}(A) \geqq \sum_{i=1}^{n} (r_i - 2) + 2.$$

LEMMA 2 (Hartfiel [2]). *If A is a nearly decomposable n-square* (0, 1)-*matrix, $n \geqq 2$, then there exist permutation matrices P and Q such that*

$$(2)\quad PAQ = \begin{bmatrix} A_1 & 0 & 0 & \cdot & \cdot & \cdot & 0 & 0 & F_1 \\ F_2 & A_2 & 0 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ 0 & F_3 & A_3 & \cdot & \cdot & \cdot & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & & \cdot & \cdot \\ \cdot & \cdot & & & & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & F_{s-1} & A_{s-1} & 0 \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 0 & F_s & A_s \end{bmatrix},$$

where $s \geqq 2$, A_1 is an n_1-square nearly decomposable matrix, $A_i = 1$, $i = 2, \cdots, n$, and each F_i has exactly one positive entry.

LEMMA 3. (Frobenius [1]). *If A is a (totally) indecomposable nonnegative matrix with spectral radius $r(A)$, then*

$$r(A) \leqq \max_i r_i,$$

and equality holds if and only if $r_1 = r_2 = \cdots = r_n$.

PROOF OF THE THEOREM. Inequality (1) was proved in [4]. We proceed to establish the condition for equality in (1). We can assume without loss of generality that $r_1 \geqq r_2 \geqq \cdots \geqq r_n$ and we prove that

$$(3)\qquad \operatorname{per}(A) = r_1$$

if and only if $r_2 = \cdots = r_n = 2$.

Suppose that (3) holds. Then, by Lemma 1,

$$\begin{aligned} r_1 &= \operatorname{per}(A) \\ &\geqq \sum_{i=1}^{n} (r_i - 2) + 2 \\ &= r_1 + \sum_{i=2}^{n} (r_i - 2), \end{aligned}$$

and, since $r_i - 2 \geqq 0$ for all i, we must have

$$r_2 = \cdots = r_n = 2.$$

Conversely, let A be a totally indecomposable (0, 1)-matrix with $r_1 \geqq r_2 = \cdots = r_n = 2$. We have to prove that this implies (3), i.e., that per $(A\ ((1\ |j)) = 1$ whenever $a_{1j} = 1$. We shall in fact establish a somewhat stronger conclusion:

(4) $$\operatorname{per}(A(1|j)) = 1, \qquad j = 1, \cdots, n.$$

We assert that it suffices to prove (4) for nearly decomposable matrices. For, suppose that A is not nearly decomposable. Then there must exist a positive entry in the first row of A, $a_{1j_1} = 1$, such that $A - E_{1j_1}$ is a totally indecomposable (0, 1)-matrix. Again, if $A - E_{1j_1}$ is not nearly decomposable, then there exists a positive entry in the first row of $A - E_{1j_1}$, $a_{1j_2} = 1$, such that $A - E_{1j_1} - E_{1j_2}$ is totally indecomposable; and so on. Thus we must finally obtain a nearly decomposable (0, 1)-matrix

$$B = A - \sum_{t=1}^{m} E_{1j_t}$$

with row sums $r_1 - m \geqq r_2 \geqq \cdots \geqq r_n = 2$. Now,

$$B(1\,|j) = A(1\,|j), \qquad j = 1, \cdots, n,$$

and therefore condition (4) is equivalent to

$$\operatorname{per}(B(1\,|j) = 1, \qquad j = 1, \cdots, n.$$

Hence we can assume without loss of generality that A is nearly decomposable with row sums $r_1 \geqq r_2 = \cdots = r_n = 2$. Let P and Q be permutation matrices such that PAQ is of the form (2), and let P and Q be so chosen that the positive entries of F_1 and F_2 lie in the $(1, n)$ and the $(n_1 + 1,\ n_1)$ positions, respectively. First note that if $r_1 = 2$, i.e., $n_1 = 1$, then PAQ is the sum of the identity matrix I_n and the permutation matrix R with ones in the subdiagonal. It is easy to check that in this case every subpermanent of $I_n + R$ of order $n - 1$ is 1.

We prove the general case by induction on n. Clearly the first row sum of PAQ is r_1, and therefore the first row sum of A_1 is $r_1 - 1$ and its other row sums (if any) are 2. Hence, by the induction hypothesis,

$$\mathrm{per}(A_1(1\,|\,j)) = 1, \qquad j = 1,\cdots,n_1. \tag{5}$$

It is easy to see from the structure of the matrix PAQ (even without any assumption on the form of A_1) that

$$\mathrm{per}((PAQ)(1\,|\,j)) = \mathrm{per}(A_1(1\,|\,j)),$$

$j = 1,\cdots,n_1$, and

$$\mathrm{per}((PAQ)(1\,|\,j)) = \mathrm{per}(A_1(1\,|\,n_1)),$$

$j = n_1 + 1,\cdots, n$. The result now follows by (5). ∎

COROLLARY. *If A is a totally indecomposable* (0,1)*-matrix with spectral radius $r(A)$, then*

$$\mathrm{per}(A) \geqq r(A), \tag{6}$$

with equality if and only if all the row sums of A are 2.

The corollary is an immediate consequence of inequality (1) and Lemma 3.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CALIFORNIA, U. S. A. 93106